MATH 2050 C Lecture 18 (Mar 18)

[Problem Set 9 posted, due on Mar 26.]

Last time ...... Sequential Criteria, Divergence criteria, & limit theorems

ASSUME: Functions are defined on $A \subseteq \mathbb{R}$, $c \in \mathbb{R}$ is a cluster pt of A.

Squeeze / Sandwich Thm (for functions)

Let $g, f, h : A \to \mathbb{R}$ be functions s.t.

$$g(x) \leq f(x) \leq h(x) \quad \forall x \in A \quad \cdots\cdots (\dagger)$$

Suppose $\lim\limits_{x \to c} g(x) = L = \lim\limits_{x \to c} h(x)$.

THEN. $\lim\limits_{x \to c} f(x) = L$.

Remarks: 1) The existence of $\lim\limits_{x \to c} f(x)$ is a conclusion

2) One only requires $(\dagger)$ to hold in some neighborhood of c.

Proof: Use sequential criteria.

Let (x_n) be a sequence in A s.t. $x_n \neq c$ $\forall n \in \mathbb{N}$. $\lim (x_n) = c$.

Claim: $\lim f(x_n) = L$

Pf: By $(\dagger)$, $g(x_n) \leq f(x_n) \leq h(x_n)$ $\forall n \in \mathbb{N}$

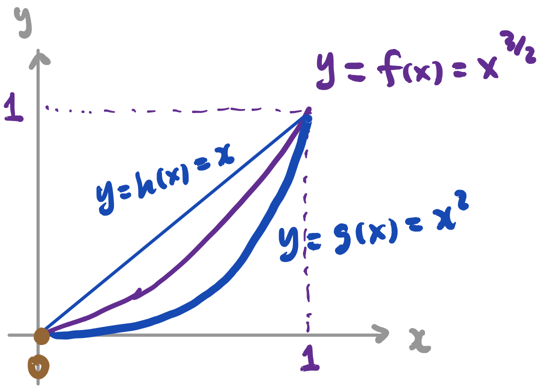
By Seq. Criteria. $\lim g(x_n) = L = \lim h(x_n)$.

By Squeeze Thm for seq., $\lim f(x_n) = L$.

$\square$

Example 1: $\boxed{\lim_{x\to 0} x^{3/2} = 0}$

Proof: Here: $f : A := \{x \in \mathbb{R} \mid x \geq 0\} \to \mathbb{R}$ where $f(x) := x^{3/2}$.



Take $g, h : A \to \mathbb{R}$ as

$g(x) = x^2$ & $h(x) = x$.

Note that

$$x^2 \leq x^{3/2} \leq x \qquad \forall x \in [0,1]$$

By squeeze thm. since

$$\lim_{x\to 0} x^2 = 0 = \lim_{x\to 0} x$$

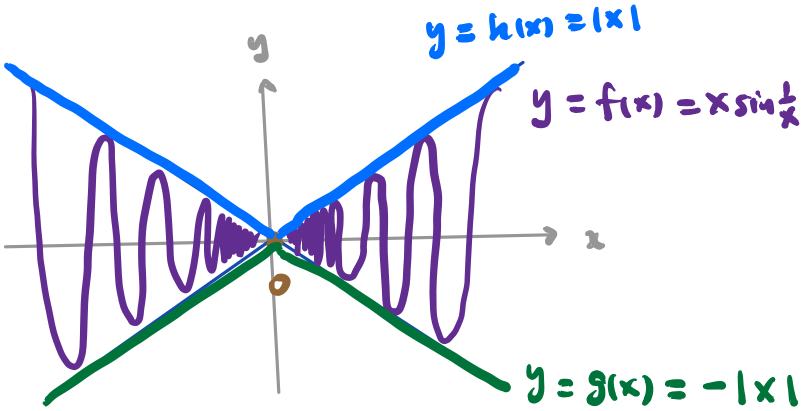
so $\lim_{x\to 0} x^{3/2} = 0$.

$\square$

Example 2: $\boxed{\lim_{x\to 0} x \sin\frac{1}{x} = 0}$

(Recall: $\lim_{x\to 0}\left(\sin\frac{1}{x}\right)$ DOES NOT EXIST by seq. criteria.)

Proof: Here: $f : A = \mathbb{R}\setminus\{0\} \to \mathbb{R}$, and $f(x) = x\sin\frac{1}{x}$.



Since $\left|\sin\frac{1}{x}\right| \leq 1$, we have

$$-|x| \leq x\sin\frac{1}{x} \leq |x| \qquad \forall x \in A$$

Now $\lim_{x\to 0} |x| = 0 = \lim_{x\to 0} -|x|$

By Squeeze Thm.

$$\lim_{x\to 0} x\sin\frac{1}{x} = 0.$$

$\square$

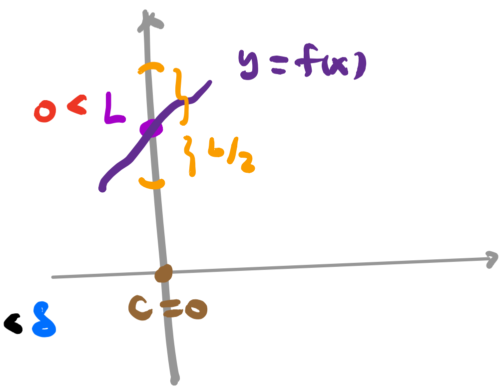
Prop: Suppose $\lim_{x \to c} f(x) = L > 0$. THEN, $\exists\, \delta > 0$ s.t.

$$f(x) > 0 \qquad \forall\, x \in A \text{ s.t. } 0 < |x - c| < \delta$$

Remark: The Prop. DOES NOT hold if we replace $>$ by $\geq$.

e.g. $L = 0$ (see Example 2 above)

Proof: Use ε-δ def$^{\underline{n}}$!



Take $\varepsilon := L/2 > 0$.

Then $\exists\, \delta = \delta\left(\frac{L}{2}\right) > 0$ s.t.

$$|f(x) - L| < \varepsilon = \frac{L}{2} \qquad \forall\, 0 < |x - c| < \delta$$

$$\Rightarrow \qquad f(x) > L - \frac{L}{2} = \frac{L}{2} > 0 \qquad \forall\, 0 < |x - c| < \delta$$

$\square$